QUATERNIONS.

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1. QUATERNIONS is a system of mathematics invented by Sir William Rowan Hamilton, an Irish Mathematician, about the year 1843. He gave to it the above name because it involves four fundamental arbitrary units, viz. VECTOR, TENSOR, VERSOR and SCALAR. These terms will be defined hereafter.

The system grew out of an attempt to geometrize the imaginaries of algebra, of the form $a\sqrt{-1}$. During the early part of the present century there were many attempts in this direction, but most of them amounted to but little more than suggestions and expressions of opinion. There was, however, one notable exception in the labors of one M. Argand, a French mathematician, who, in 1806, published an essay on the manner of representing Imaginary Quantities, whose methods were substantially reproduced by a Mr. Warren, in England in 1828, and also during the same year by one M. Mourey, in Paris. These writers produced some algebraic expressions which are substantially the same as those used by Hamilton—also their modulus corresponds with Hamilton's tensor, and their *verser* is identical with his *versor* (the former being the French and the latter the English of the same word). Their method of multiplying together two directed lines in a plane—which consists in multiplying together their modulii and adding their versers—is apparently the same as Hamilton's. But Hamilton's fundamental conceptions of methods were evidently different from those of his predecessors and contemporaries. Thus, M. Argand's operations for multiplying two directed lines were in one plane, producing a third line in the same plane; but Hamilton's method produces a third line perpendicular to the plane of the two lines, and hence the operation involves the three rectangular dimensions of space. It is worthy of remark, that no method of treating imaginaries except Hamilton's, grew to a system; but this, by the hands of the inventor was so developed as to make a complete system of geometry, including its application to trigonometry and mechanics.

One of the earliest statements on record, in regard to the geometrical signification of the even roots of negative numbers, was by Dr. Wallis of Oxford, in his 'Treatise of Algebra', published in 1685; in which he states that *positive* implies one direction along a line, *negative* the opposite direction along the same line, but the $\sqrt{-a}$ cannot be found in the line but above the line, it may be in the same plane.

In the *Philosophical Transactions* (London) for 1806 is a paper by M. Abbe' Buée, in which the writer maintained that the $\sqrt{-1}$, as a symbol, denoted perpendicularity—a view which had been suggested by others, but no system ever grew out of it.

The negatives and imaginaries of algebra are not considered as quantities*; they are, primarily, symbols of unexecuted operations, but when they appear in the result of an analytical process, they are to be interpreted, and the interpretation sometimes leads to an enlargement of the language of the problem, and in other cases to a limitation of it. For instance, if the language implies that the results are to be all positive, but a solution shows that there are several sets of results, some of which are positive and others negative, it shows that the language need not be taken literally. Similarly, those conditions which would produce imaginary results, seem to limit the problem; as for instance in the equation to the circle, those values of the coordinates which give imaginary results do not belong to the locus.

The *positives* and *negatives* of algebra are readily geometrized, and since certain real quantities may be represented in forms involving imaginaries, it is natural to seek a geometrical expression for them. Such is the case with Euler's formulas—

$$2\sqrt{(-1)}\sin x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2Shx,$$

$$2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2Chx,$$

(*Shx* and *Chx* being the modern notation) where $\sin x$ and $\cos x$ are real for all values of x.

The $\sqrt{-1}$ when deduced according to Hamilton's system, has, in that system, a real geometrical signification, and can be conceived of as definitely as an are, or angle, or any other geometrical magnitude; yet his system does not pretend to geometrize the $\sqrt{-1}$ of algebra. It leaves that symbol, when it is the result of an algebraic operation, as imaginary as ever, as unreal as ever, as ungeometrical as ever, and as Hamilton says† it "represents no real line". I have emphasized this fact, because we occasionally meet with writers of recent date who boldly assert that the $\sqrt{-1}$ of algebra is real and may be represented by a geometrical line; although they are compelled to admit that it is impossible to find a quantity whose square is negative. It is not improbable that many have been mislead by the identity of the symbol $\sqrt{-1}$ in the two systems, so that while they see its real character in quaternions, they do not understand why its application is not

*Hamilton's Lectures on Quaternions, preface, p. 2.

†Hamilton's Lectures, p. 638.

universal; but the expression is the result of very different operations in the two systems. Modern mathematics has given new meanings to old symbols, or, more properly speaking, it has used old symbols to represent new ideas, and thus the same symbol has a double meaning — and in some cases several meanings. Thus, in algebra, dx implies that x is multiplied by d, but in calculus d implies an operation; in the expression d^2x, the figure 2 may imply a square or a second operation; —1 in the expression a^{-1} implies a reciprocal, but in the expression $\sin^{-1}x$ an inverse function; the period (.) is used to denote a decimal fraction, multiplication, or a fluxion; $\beta = 0$ in algebra means that β is zero, but in modern geometry it may represent the equation of a line; &c. Conversely, the same symbol may be put under different forms; thus we have

$$a \div b = \frac{a}{b};\ a^{-1} = \frac{1}{a};\ dx = x;\ \sqrt{-1} = (-1)^{\frac{1}{2}};\ \&c.$$

Hamilton's system of quaternions and the imaginaries of algebra have a relation to each other similar to that between Chemistry and Alchemy. The false science of alchemy gave birth to the true science of chemistry,* but the latter, instead of solving the problem of the former—the transmutation of metals into gold—shows it to be all the more impossible, and has entered new fields of inquiry; so the imaginaries of algebra gave rise to the real system of Quaternions, but instead of geometrizing the former leaves them as imaginary as before and enters its own field of inquiry by developing new processes of analysis.

As soon as chemistry was established there was a deluge of pretenders in science who *attempted* to find the philosopher's stone; so after the establishment of the science of quaternions, there were, and still are, some who hold that *all* imaginaries are so only in name, but are in fact real quantities.

(To be continued.)

DETERMINATION OF A SPHERE WHICH CUTS FIVE GIVEN SPHERES AT THE SAME ANGLE.

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General Benjamin Alvord a few days ago attracted my attention to this problem as being one which had, at one time, possessed a peculiar interest

*J. Andrew, in the ninth edition of the Encyclopædia Britanica, says, "Alchemy was, we may say, the sickly but imaginative infancy through which modern chemistry had to pass before it attained its majority."